

Bernoulli-Taylor formula in the case of Q -umbral Calculus

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Abstract

In this note we derive the Q -difference Bernoulli-Taylor formula with the rest term of the Cauchy form by the Viskov's method [1]. This is an extension of technique presented in [5, 6, 7] by the use of Q -extended Kwaśniewski's $*_{\psi}$ -product [3, 4]. The main theorems of Q -umbral calculus were given by G. Markowsky in 1968 (see [2]) and extended by A.K.Kwaśniewski [3, 4].

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1 Introduction - Q -umbral calculus

We shall denote by \mathbf{P} the algebra of polynomials over the field \mathbf{F} of characteristic zero.

Let us consider a one parameter family F of sequences. Then a sequence ψ is called admissible ([3, 4]) if $\psi \in \mathcal{F}$. Where

$$\mathcal{F} = \{\Psi : \mathbf{R} \supset [a, b]; q \in [a, b] : \Psi(q) : \mathbf{Z} \rightarrow \mathbf{F}; \Psi_0(q) = 1, \Psi_n(q) \neq 0, \\ \Psi_{-n}(q) = 0, n \in \mathbf{N}\}$$

Now let us to introduce the Ψ -notation [3, 4]:

$$n_{\psi} = \Psi_{n-1}(q)\Psi_n^{-1}(q); \\ n_{\psi}! = n_{\psi}(n-1)_{\psi} \cdots 2_{\psi}1_{\psi} = \Psi_n^{-1}(q), \\ n_{\psi}^k = n_{\psi}(n-1)_{\psi} \cdots (n-k+1)_{\psi}, \\ \binom{n}{k}_{\psi} = \frac{n_{\psi}^k}{k_{\psi}!}, \\ exp_{\psi}\{y\} = \sum_{k=0}^{\infty} \frac{y^k}{k_{\psi}!}$$

Definition 1.1. [2, 3] Let Q be a linear map $Q : \mathbf{P} \rightarrow \mathbf{P}$ such that:

$$\forall p \in \mathbf{P} \quad deg(Qp) = (deg p) - 1$$

$deg p = -1$ means $p = const = 0$. Then Q is called a generalized differential operator [2].

Definition 1.2. [2] A normal sequence of polynomials $\{q_n\}_{n \geq 0}$ has the following properties:

- (a) $\deg q_n(x) = n$;
- (b) $\forall x \in \mathbf{F} \quad q_0(x) = 1$;
- (c) $\forall n \geq 1 \quad q_n(0) = 0$.

Definition 1.3. [2, 3] Let $q_n(x)_{n \geq 0}$ be the normal sequence. Then we call it the ψ -basic sequence of the generalized differential operator Q (or $Q - \psi$ -basic sequence) if:

$$\forall n \geq 0 \quad Qq_n(x) = n_\psi q_{n-1}(x)$$

In [2] it is shown that once a differential operator Q is given a unique ψ -basic polynomial sequence and the other way round: given a normal sequence $q_n(x)_{n \geq 0}$ there exists a uniquely determined generalized differential operator Q .

Definition 1.4. [2, 3] The \hat{x}_Q -operator (Q -multiplication operator, the operator dual to Q) is the linear map $\hat{x}_Q : \mathbf{P} \rightarrow \mathbf{P}$ such that:

$$\forall n \geq 0 \quad \hat{x}_Q q_n(x) = \frac{n+1}{(n+1)_\psi} q_{n+1}(x)$$

Note that: $[Q, \hat{x}_Q] = id$

Definition 1.5. [3] Let $q_n(x)_{n \geq 0}$ be a $Q - \psi$ -basic sequence. Let

$$E_q^y(Q) = E^y(Q) = \exp_{Q, \psi} y Q = \sum_{k=0}^{\infty} \frac{q_k(y) Q^k}{k_\psi!}$$

$E_q^y(Q) = E^y(Q)$ is called the $Q - \psi$ -generalized translation operator.

As was annouced in [5, 7], the notion of Kwaśniewski's $*_\psi$ product and its properties presented in [3] can be easily Q -exteted as follows.

Definition 1.6. [3]

$$x *_Q q_n(x) = \hat{x}_Q(q_n(x)) = \frac{n+1}{(n+1)_\psi} q_{n+1}(x), \quad n \geq 0$$

$$x^n *_Q q_n(x) = (\hat{x}_Q^n)(q_1(x)) = \frac{(n+1)!}{(n+1)_\psi!} q_{n+1}(x), \quad n \geq 0$$

Therefore

$$x *_Q \alpha 1 = x *_Q \alpha q_0(x) = \hat{x}_Q(\alpha q_0(x)) = \alpha \hat{x}_Q(q_0(x)) = \alpha x *_Q 1$$

and

$$f(x) *_Q q_n(x) = f(\hat{x}_Q) q_n(x)$$

for every formal series f.

Definition 1.7. According to definition above and [3] we can define Q -powers of x by recurrence relation:

$$\begin{aligned} x^{0*Q} &= 1 = q_0(x) \\ x^{n*Q} &= x *_Q (x^{(n-1)*Q}) = \hat{x}_Q(x^{(n-1)*Q}) \end{aligned}$$

It is easy to show that:

$$x^{n*Q} = x *_Q x *_Q \dots *_Q 1 = \frac{n!}{n_\psi!} q_n(x), \quad n \geq 0$$

Also note that:

$$x^{n*Q} *_Q x^{k*Q} = \frac{n!}{n_\psi!} q_{k+n}(x)$$

and

$$x^{k*Q} *_Q x^{n*Q} = \frac{k!}{k_\psi!} q_{k+n}(x)$$

so in general i.e. for arbitrary admissible ψ and for every $\{q_n\}_{n \geq 0}$ it is noncommutative.

Due to above definition one can proof the following Q -extended properties of Kwaśniewski's $*_\psi$ product [3, 4].

Proposition 1.8. Let f, g be formal series. Then for $*_Q$ defined above holds:

- (a) $Qx^{n*Q} = nx^{(n-1)*Q}, \quad n \geq 0;$
- (b) $\exp_{Q,\psi}[\alpha x] = \exp \alpha \hat{x}_Q 1$ where $\exp_{Q,\psi} \alpha x = \sum_{k \geq 0} \frac{q_k(x) \alpha^k}{k_\psi!};$
- (c) $Q(x^k *_Q x^{n*Q}) = (Dx^k) *_Q x^{n*Q} + x^k *_Q (Qx^{n*Q});$
- (d) $Q(f *_Q g) = (Df) *_Q g + f *_Q (Qg),$ (Q -Leibnitz rule);
- (e) $f(\hat{x}_Q)g(\hat{x}_Q)1 = f(x) *_Q \tilde{g}; \quad \tilde{g}(x) = g(\hat{x}_Q)1.$

According to [2, 3], let us to define Q -integration operator which is a right inverse operation to generalized differential operator Q i.e.:

$$Q \circ \int d_Q t = id$$

Definition 1.9. We define Q -integral as a linear operator such that

$$\int q_n(x) d_Q x = \frac{1}{(n+1)_\psi} q_{n+1}(x); \quad n \geq 0.$$

Proposition 1.10.

- (a) $Q \circ \int_{\alpha}^x f(t) d_Q t = f(x)$;
- (b) $\int_{\alpha}^x (Qf)(t) d_Q t = f(x) - f(\alpha)$;
- (c) formula for integration "per partes" :

$$\int_{\alpha}^{\beta} (f *_Q Qg)(x) d_Q x = [(f *_Q g)(x)]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} ((Df) *_Q g)(x) d_Q x.$$

2 Q -umbral calculus Bernoulli-Taylor formula

In [1] O.V.Viskov establishes the following identity

$$(2.1) \quad \hat{p} \sum_{k=0}^n \frac{(-\hat{q})^k \hat{p}^k}{k!} = \frac{(-\hat{q})^n \hat{p}^{n+1}}{n!}.$$

what he calles the Bernoulli identity. Here \hat{p}, \hat{q} stand for linear operators satysfying condition:

$$(2.2) \quad [\hat{p}, \hat{q}] = \hat{p}\hat{q} - \hat{q}\hat{p} = id$$

Now let \hat{p} and \hat{q} be as below:

$$\hat{p} = Q, \quad \hat{q} = \hat{x}_Q - y, \quad y \in \mathbf{F}$$

From definition (1.4) we have $[Q, \hat{x}_Q - y] = id$. After submission into (2.1) we get:

$$Q \sum_{k=0}^n \frac{(y - \hat{x}_Q)^k Q^k}{k!} = \frac{(y - \hat{x}_Q)^n Q^{n+1}}{n!}$$

Now let us apply it to any polynomial (formal series) f :

$$Q \sum_{k=0}^n \frac{(y - \hat{x}_Q)^k (Q^k f)(t)}{k!} = \frac{(y - \hat{x}_Q)^n (Q^{n+1} f)(t)}{n!}.$$

Now using definitions (1.6) and (1.7) of $*_Q$ -product we get:

$$Q \sum_{k=0}^n \frac{(y - x)^{k*_Q} *_Q (Q^k f)(t)}{k!} = \frac{(y - x)^{n*_Q} *_Q (Q^{n+1} f)(t)}{n!}.$$

After integration $\int_y^x d_Q t$ using proposition (1.2) it gives Q -difference calculus Bernoulli-Taylor formula of the form:

$$(2.3) \quad f(x) = \sum_{k=0}^n \frac{1}{k!} (x - y)^{k*_Q} *_Q (Q^k f)(y) + R_{n+1}(x)$$

where R_{n+1} stands for the rest term of the Cauchy type :

$$(2.4) \quad R_{n+1}(x) = \frac{1}{n!} \int_y^x (x - t)^{n*_Q} *_Q (Q^{n+1} f)(t) d_{\psi} t.$$

3 Special cases

1. An example of generalized differential operator is $Q \equiv \partial_\psi$. Then $q_n = x^n$ and $\partial_\psi x^n = n_\psi x^{n-1}$ for an admissible ψ . ∂_ψ is called ψ -derivative. Then also $\hat{x}_Q \equiv \hat{x}_\psi$ and $\hat{x}_\psi x^n = \frac{(n+1)}{(n+1)_\psi} x^{n+1}$ and $[\partial_\psi, \hat{x}_\psi] = id$. In this case we get ∂_ψ -difference calculus Bernoulli-Taylor formula presented in [5, 6, 7] of the form

$$f(x) = \sum_{k=0}^n \frac{1}{k!} (x - \alpha)^{k*_\psi} *_\psi (\partial_\psi^k f)(\alpha) + R_{n+1}(x)$$

with

$$R_{n+1}(x) = \frac{1}{n!} \int_\alpha^x (x - t)^{n*_\psi} *_\psi (\partial_\psi^{n+1} f)(t) d_\psi t.$$

In [3] there is given a condition for the case $Q = Q(\partial_\psi)$ for some admissible ψ (see Section 2, Observation 2.1).

2. For $Q = \partial_\psi$, $q_n(x) = x^n$, $n \geq 0$ the choice $\psi_n(q) = \frac{1}{[R(q^n)]!}$, $R(x) = \frac{1-x}{1-q}$ gives the well known q -factorial $n_q! = n_q(n-1)_q \dots 2_q 1_q$, for $n_q = 1 + q + q^2 + \dots + q^{n-1}$. Then $\partial_\psi = \partial_q$ becomes the well known Jackson's derivative ∂_q :

$$(\partial_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

The ∂_q -difference version of the Bernoulli-Taylor formula was given in [5] by the use of $*_q$ -product.

3. By the choice $Q \equiv D \equiv \frac{d}{dx}$, $q_n(x) = x^n$ and $\psi_n = \frac{1}{n!}$ after submission to (2.3), (2.4) we get the classical Bernoulli-Taylor formula of the form:

$$f(x) = \sum_{k=0}^n \frac{(x - \alpha)^k}{k!} f^{(k)}(\alpha) + \int_\alpha^x \frac{(x - t)^n}{n!} f^{n+1}(t) dt$$

where $f^{(k)}(\alpha) = (D^k f)(\alpha)$.

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